Complex Analysis: Midterm Exam

Aletta Jacobshal 01, Monday 14 December 2015, 09:00 – 11:00
Exam duration: 2 hours

Instructions — read carefully before starting

- Do not forget to very clearly write your **full name** and **student number** on each answer sheet and on the envelope. Do not seal the ennvelope.
- 10 points are "free" for handing-in the assignment. There are 5 questions and the total number of points is 100. The exam grade is the total number of points divided by 10.
- Solutions should be complete and clearly present your reasoning.

Question 1 (20 points)

Consider the function

$$f(z) = \frac{y-1}{x^2+(y-1)^2} + i\,\frac{x}{x^2+(y-1)^2},$$

where $z = x + iy$.

(a) (10 points) Prove that $f(z)$ is analytic for $z \neq i$.

(b) (10 points) Write $f(z)$ as a function of z.

Question 2 (20 points)

Consider the two smooth arcs γ_1 and γ_2 shown in Figure 1. The arc γ_1 is a straight line from the point $-i$ to the point $1+i$ and the arc γ_2 is a smooth curve from $1+i$ to 2 for which $y = (x-2)^2$.

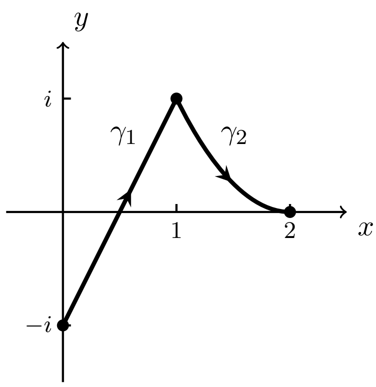

Figure 1: Smooth arcs γ_1 and γ_2 for Question 2.

(a) (10 points) Parameterize the two smooth arcs.

(b) (10 points) Compute the integral

$$\int_{\gamma_1} \bar{z}^2\, dz.$$

Question 3 (20 points)

Compute the value of the integral

$$\int_\Gamma \frac{\cos(\pi z)}{(z-i)(z-3)^2}\,dz$$

where Γ is the closed contour shown in Figure 2. Give the result as a complex number in Cartesian form $x + iy$.

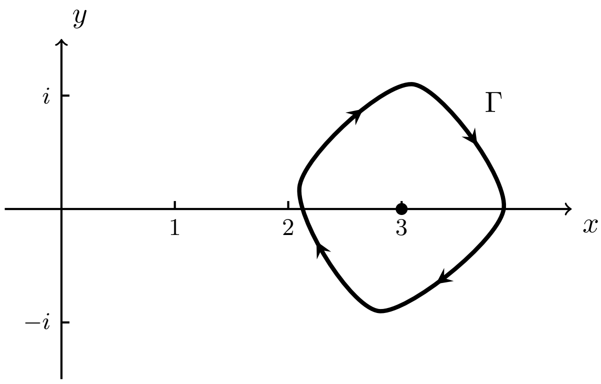

Figure 2: Contour Γ for Question 3.

Question 4 (20 points)

Consider the function

$$f(z) = e^{\frac{1}{2}\,\mathrm{Log}(1-z)}\,e^{\frac{1}{2}\mathcal{L}_0(1+z)},$$

where $\mathcal{L}_0(z)$ is the branch of the logarithm function with a branch cut along the positive real axis $[0,\infty)$ and $\mathrm{Log}(z)$ is the principal value of the logarithm. Show that $f(z)$ is continuous at z in the interval $(1,\infty)$ on the real axis.

Question 5 (10 points)

Consider an entire function $f(z)$ such that its absolute value is bounded below by a positive number, that is, $|f(z)| > M > 0$ for all $z \in \mathbb{C}$. Show that $f(z)$ is constant.

End of the exam (Total: 90 points)